EXPLORATION OF FOUR ANTI-FOUNDATION AXIOMS

by

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LIST OF ABBREVIATIONS

Abbreviation	**Description**
AFA	The Anti-Foundation Axiom
apg	Accessible pointed graph
BAFA	Boffa's Anti-Foundation Axiom
FAFA	Finsler's Anti-Foundation Axiom
SAFA	Scott's Anti-Foundation Axiom
ZFC	Zermelo-Fraenkel Set Theory with Choice
$\mathrm{ZFC}^{-\mathrm{f}}$	Zermelo-Fraenkel Set Theory with Choice without Foundation

1 INTRODUCTION

Zermelo-Fraenkel Set Theory with the Axiom of Choice (ZFC) is the standard axiomatic set theory and includes the Axiom of Foundation, which states that every non-empty set must contain an element disjoint from the set itself. As a result, ZFC forbids sets that have infinite descending membership chains $x_0 \ni x_1 \ni x_2 \ni \ldots$, such as sets with closed membership loops $x_0 \ni x_1 \ni \ldots \ni x_n = x_0$ or which contain themselves as elements. These sets are called "non-well-founded" and are not considered in standard set theory.

However, it is possible to modify ZFC by removing Foundation and allowing for the existence of non-well-founded sets. When doing so, the system ZFC without Foundation proves to be insufficient to characterize which non-well-founded sets exist, and so alternative axioms that violate Foundation are desired. The work of Peter Aczel in [1] is largely devoted to exploring four such alternative axioms, which are the primary axioms considered in non-well-founded set theory. The replacement of Foundation with any one of these four axioms produces a system which is proven to be relatively consistent with ZFC.

This thesis will present these axioms using a graph-theoretic approach as developed by Aczel and others. It will provide exposition on the hierarchy formed by these axioms in which the set-theoretic universe of the most restrictive is a subset of the next, and so on. It will also provide particular focus on Aczel's most permissive axiom, Boffa's Anti-Foundation Axiom.

Chapter 2 will cover relevant background set theory and Chapter 3 will introduce relevant graph-theoretic notation. Chapter 4 will cover three alternative axioms to Foundation, while Boffa's Anti-Foundation Axiom is discussed in Chapter 5. Chapter 6 will briefly compare these axioms and we will conclude with an exploration of Rieger's Theorem in Chapter 7.

2 BACKGROUND

2.1 Preliminaries

Set theory is the formal study of collections of mathematical objects, called “sets”. Early in its inception, set theory was primarily employed to answer questions about infinity. Georg Cantor is credited as the father of set theory, which he utilized in 1873 to differentiate between *countable* and *uncountable* sizes of infinity [11].

More recently, set theory has served as a formal foundation for broader mathematics. Using a base set of axioms, mathematical objects (integers, functions, topological spaces, etc.) are encoded as sets in order to ensure mutual compatibility and shared standard between different disciplines [12]. If one wishes to prove a fact about the expansive general mathematical world, one potential strategy is to encode mathematical objects as sets, and then prove a fact about set theory. Additionally, if mathematicians working in disparate fields agree to start from the same base set theoretic assumptions, it ensures that their constructions and theorems are mutually compatible, allowing the diverse areas of mathematics to talk to each other.

2.2 ZFC Set Theory

Zermelo-Fraenkel Set Theory with the Axiom of Choice (ZFC) has emerged as the standard formulation of set theory. Borrowing heavily from [9], the axioms of ZFC can be presented as follows:

✧**Extensionality:** If X and Y are sets that contain the same elements, then $X = Y$.

✧**Pairing:** For any sets X and Y there exists a set $\{X, Y\}$ containing precisely X and Y.

✧**Schema of Comprehension:** If P is a property, then for any set X there exists a set $Y = \{u \in X \mid P(u)\}$ which is the set of all $u \in X$ satisfying property P. Comprehension is also referred to as the axiom schema of Separation.

✧**Union:** If X is a set of sets then there exists a set $\cup X$, the set of all elements of all sets in X. That is $\cup X = \{u \in w \mid w \in X\}$.

✧**Power Set:** For any set X there exists a set $\mathscr{P}(X)$ which is the set of all subsets of X.

✧**Empty:** There exists a set $\varnothing$ which has no elements.

✧**Infinity:** There exists an infinite set. This set is often taken to be the natural numbers, written as ω.

✧**Schema of Replacement:** If F is a formula defining a function and X is a set, then the image of X by F is a set. That is there exists a set $F(X) = \{F(x) \mid x \in X\}$.

✧**Foundation:** If X is a nonempty set then X must contain an element m so that X and m are disjoint. Foundation is also referred to as the axiom of Regularity.

✧**Choice:** If X is a set of sets then there exists a function F which chooses a single element from each set in X. That is if $A \in X$ then $F(A) \in A$. [1]

The axioms of ZFC can be roughly grouped according to the roles they play in the larger system.

✩**Table 2.2.1:** The axioms of ZFC grouped by role. Note that Foundation is the only axiom that imposes an explicit restriction on which collections are considered sets.

Identity: Tell us when two sets are equal	**Construction:** Make new sets from given sets.	**Existence:** Assert the existence of some set.	**Restriction:** Assert a certain type of set does not exist.	**Choice:** Similar to construction, but non-explicit and non-unique.
Extensionality	Pair Comprehension Union Powerset Replacement	Empty Infinity	Foundation	Choice

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[1] In ZFC, Choice is equivalent to the statement that any set has a well-ordering, a notion we will define in the following section.

Though these are the standard axioms of set theory, there is often good reason (beside simply mathematical curiosity) to consider alternative axiom systems which modify, weaken, or strengthen the axioms above. This paper will explore alternative axiom systems which omit or explicitly violate the axiom of Foundation.

2.3 The Axiom of Foundation

Before we consider axiom systems that violate Foundation, let's spend some time understanding this axiom and its purpose.

Recall that Foundation asserts that if X is a nonempty set then X contains an element m so that $X \cap m = \varnothing$. Such an $m \in X$ is a $\in$-minimal element of X. The following definition will help us state an equivalent (and often more intuitive) formulation of Foundation.

✿**Definition 2.3.1:** If $x_0, x_1, x_2, \ldots$ is an infinite sequence of sets so that

$$x_0 \ni x_1 \ni \ldots \ni x_n \ni x_{n+1} \ni \ldots$$

we say that such a sequence is an infinite descending membership chain.

In the presence of the other ZFC axioms, the following equivalence can be shown.

✧**Theorem 2.3.2** Foundation is equivalent to the assertion there is no infinite descending membership chain. For reference, see [9].

Proof. Assume by way of contrapositive that there exists an infinite descending membership chain $x_{n+1} \in x_n$ for each $n \in \omega$. Since this chain of sets is indexed by the natural numbers, we can naturally define a function f with domain ω so that $f(n) = x_n$. By Replacement, the collection $X = \{x_n \mid n \in \omega\}$ is then a set.

We then establish that X has no $\in$-minimal element. Given any $m \in X$, it must be that $m = x_n$ for some n, and so $x_{n+1} \in (m \cap X)$, meaning $m \cap X \neq \varnothing$. Thus Foundation implies there is no infinite descending membership chain.

For the other implication, assume again by way of contrapositive that there exists some nonempty set X so that for all $m \in X$ we have $m \cap X \neq \varnothing$. We'll construct an infinite descending membership chain recursively.

Since X is non-empty, it contains some element x_0. By assumption $x_0 \cap X \neq \varnothing$, and so we may choose an element x_1 with both $x_1 \in x_0$ and $x_1 \in X$.

Recursively, given x_n we know that $x_n \cap X \neq \varnothing$ and thus can choose an element x_{n+1} with $x_{n+1} \in x_n$ and $x_n \in X$. By the Axiom of Choice we may make countably infinite-many such choices and construct the infinite descending membership chain $x_0 \ni x_1 \ni x_2 \ni \dots$ as desired. ■

Now that we've shown the above equivalence, the following corollaries are almost immediate.

✧**Corollary 2.3.3:** Foundation implies that no set X is an element of itself.

If there existed a set X with $X \in X$, this would admit the infinite descending membership chain $X \ni X \ni X \ni \dots$, violating Foundation.

✧**Corollary 2.3.4:** Foundation implies that there is no closed membership loop

$$x_0 \ni x_1 \ni \dots \ni x_n = x_0.$$

Similar to the above, the existence of such a loop would admit the infinite descending membership chain $x_0 \ni \dots \ni x_{n-1} \ni x_0 \ni \dots \ni x_{n-1} \ni \dots$, again violating Foundation.

One of the biggest advantages to utilizing Foundation is that it allows every set to be assigned a rank[2] in what is known as the cumulative hierarchy of sets. The following definitions will help us specify exactly what this means.

Definition 2.3.5: A relation $<$ on a set A is a linear ordering when it satisfies the following:

- $a < b < c$ implies $a < c$.
- For all $a, b \in A$ exactly one of the following hold: either $a < b$, $a > b$, or $a = b$.

Definition 2.3.6: Further, $<$ is said to be a well-ordering when it is both linear and every non-empty subset of A has a minimal element; that is for every $B \subseteq A$ there exists $b \in B$ so that $b \leq b'$ for all $b' \in B$.

Definition 2.3.7: A set T is said to be transitive when $x \in y \in T$ implies $x \in T$.

Definition 2.3.8: A set α is an ordinal when both the following hold:

1. α is transitive
2. The set membership relation $\in$ is a well-ordering on α.

The collection of all ordinals is denoted as Ord.

It's not too difficult to show that if α is an ordinal then the set $\alpha + 1 := \alpha \cup \{\alpha\}$ is also an ordinal, called the successor of α. If α is the successor of some other ordinal, we say α is a successor ordinal. However, there are some ordinals which are not the successor of any ordinal. Such ordinals are called limit ordinals.

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[2] Universes of sets which violate Foundation can still admit notions of rank, but they are typically less straightforward than the one provided by Foundation. See [18].

Essentially, ordinals are sets with the desirable property of transitivity and which represent well-orderings. In fact, it can be shown that any well-ordering on any set is isomorphic to some ordinal α. Ordinals can also be thought of the set-theoretic extension of the natural numbers; in set theory every finite natural number $0, 1, 2, \ldots$ is an ordinal, as well as the entire collection of natural numbers ω, followed by the ordinal $\omega + 1$, $\omega + 2$, and so on.

One of the many applications of ordinals in set theory is to define the following notion.

❀**Definition 2.3.9:** For any ordinal α, we define the set V_α recursively as follows:

$$V_0 = \varnothing$$

$$V_{\alpha+1} = \mathscr{P}(V_\alpha)$$

$$V_\alpha = \cup\{V_\gamma \mid \gamma \in \alpha\} \text{ for limit } \alpha$$

By definition these sets form a linear hierarchy where if $\alpha \in \beta$ implies $V_\alpha \subseteq V_\beta$. Thus we can assign to each set x an ordinal rank(x), a least ordinal α so that $x \in V_{\alpha+1}$.

✧**Theorem 2.3.10:** Foundation implies that every set has a rank. That is, for each set x there is a least ordinal α so that $x \in V_\alpha$. Again for reference see [9].

Proof. Since the class of ordinals itself is well-ordered by $\in$, if $x \in V_\alpha$ for some ordinal α then there is a least such ordinal. Therefore it suffices to show that $x \in V_\alpha$ for any ordinal α.

For sake of contradiction suppose that there exists a set x which does not appear in V_α for any ordinal α. We note that if every element of x were in some V_α, we could define a function f so that for each $y \in x$ we have $f(y)$ is the minimum ordinal α so that $y \in V_\alpha$. Then the image of f is a set of ordinals, and we can thus take the supremum of this set of ordinals to produce an ordinal γ so that each $y \in V_\gamma$. In other words, we define the ordinal γ by

$$\gamma = \sup\{\alpha \in \mathrm{Ord} \mid \alpha \text{ is the minimal ordinal such that } y \in V_\alpha \text{ for some } y \in x\}$$

Then x would be in $V_{\gamma+1}$. For notational convenience, we'll call a set A hyper when $A \notin V_\alpha$ for any $\alpha \in \mathrm{Ord}$. So, we've just shown that any set A which is hyper must contain an element which is also hyper.

By Choice, we know that any set can be well ordered. Define recursively

$$x_0 = x$$

$$x_{n+1} = \text{the least } y \in x_n \text{ such that } y \text{ is hyper,}$$

where "least" is in respect to the well-ordering on x_n that Choice guarantees exists. This provides an infinite membership chain $x_0 \ni x_1 \ni x_2 \ni \ldots$, which contradicts Foundation. Specifically, the set $\{x_n \mid n \in \omega\}$ would have no $\in$-least element. ■

Therefore, Foundation provides a linear (and in fact well-ordered) hierarchy to the universe of sets, a property which has many useful applications. However, there could be situations where forgoing this linear hierarchy (and thus forgoing Foundation) could be useful or desirable.

2.4 A Historical Note on Paradox and Foundation

ZFC set theory (as well as axiomatic set theory generally) is predated by what has since become known as naive set theory. In naive set theory, any definable property P provides for the existence of a set $X = \{x \mid P(x)\}$, the set of all objects with property P. For example, one may define and consider “the set of all positive real numbers”, or “the set of all people in New York City”, or “the set of all sets” in naive set theory.

Though naive set theory may seem at first a straightforward and intuitive approach, it gives rise to contradiction. Building on work by Burali-Forti in [4], Russell famously discussed in [5] the set $X = \{y \mid y \notin y\}$, a set definable by the property $P(y) =$“y is not in y” and thus a valid set in naive set theory. One then may ask if X itself is in X. If so, then by definition X satisfies $P(X)$, meaning $X \notin X$, a contradiction. Similarly if $X \notin X$ then it satisfies $P(X)$ and so it must be that $X \in X$, again a contradiction. Since we reach contradiction in either case, we conclude that naive set theory is an inconsistent and therefore ineffective theory.

Following Russell’s discovery, axiomatic set theory (and eventually ZFC) emerged as a formal approach to set theory which avoided Russell’s and similar paradoxes by restricting the universe of sets. ZFC as presented above provides outright for the existence of only two sets: namely $\varnothing$ and ω. For a collection to be considered a valid set in ZFC, it must be constructed from these two starting points by iteratively applying the other axioms (Pair, Union, Powerset, Comprehension, etc.). By restricting the collections that we admit as sets in this way, we prevent the construction of Russell’s set $X = \{y \mid y \notin y\}$ and (to the best of our knowledge) avoid paradox.

Rieger notes in [14] that there may be an inaccurate conception held about Russell’s paradox which sees it as arising from allowing self-containing sets. Some may then erroneously believe that Foundation is justified on the grounds that it prevents Russell’s paradox by outlawing such sets. However, the axiom of Foundation is neither necessary nor sufficient to prevent Russell-like paradoxes. In later sections, we’ll see axiomatic set theories which admit self-containing sets and are consistent (assuming ZFC is also consistent). Thus Foundation, though it has its applications, is not strictly required for a consistent set theory [1].

On the other hand, any set theory which admits the naive comprehension schema (the assertion that any property defines a valid set) will be susceptible to Russell-like paradoxes regardless of its stance on self-containing sets. In ZFC, the heavy lifting of paradox prevention is done by Comprehension; in general one may not assert that the collection of *all* y satisfying $P(y)$ forms a set, but only that the *subset* of some pre-existing set A consisting of all $y \in A$ satisfying $P(y)$ is a set [14].

Rieger argues in [14] that "when forced by paradox to reform a naive concept, [one should] preserve as much of it as possible". As we'll see in later sections, the Axiom of Foundation restricts the universe of sets more than is necessary to avoid paradox.

2.5 Models of Set Theory

We will occasionally discuss models of particular collections of set theory axioms, and so we'll say a bit about what it means for something to be a "model" of a set theory. Readers who are already comfortable with basic model theory are invited to skip this subchapter.

First, let φ be a sentence in the language of set theory. Informally, φ is a statement utilizing the standard first-order logic symbols ($\forall, \neg, \wedge, \Rightarrow$, etc) along the binary relation-symbol $\in$ which will be used to represent set membership. A sentence φ is also required to have no free variables, meaning any variables utilized in φ are attached to either a $\exists$ or $\forall$ quantifier.

❏ **Example 2.5.1:** A sentence in the language of set theory might be:

$$\exists x \forall y(\neg(y \in x)).$$

With the usual interpretation, this sentence can be read as “there exists a set x so that for all y, it is not the case that y is in x” (that is, the Empty Set Axiom). We note this sentence has no free variables, since x is "bound" by the $\exists$ quantifier and y is bound by the $\forall$ quantifier.

We note that all of the axioms of ZFC can be expressed as sentences in the language of set theory. For a more thorough discussion of formulas and sentences in the language of set theory, see [6].

❀**Definition 2.5.2:** Let M be a class (an arbitrary collection) and let ε be a binary relation on M. Let φ be a sentence in the language of set theory. We then say that (M, ε) models φ, often written $M \vDash \varphi$, if replacing every $\in$ in φ with ε and relativizing all the quantifiers in φ (that is, replacing each “$\forall x$” with “$\forall x \in M$” and each “$\exists x$” with “$\exists x \in M$”) produces a true statement about the elements of M.

❑ **Example 2.5.3:** Consider the pair $(\mathbf{N}, <)$ where $\mathbf{N}$ is the natural numbers (including zero) and $<$ is the standard less-than relation on $\mathbf{N}$. Let φ be the Empty Set Axiom as described in the previous example. To see if $M \vDash \varphi$, we'll see what happens when we relativize our quantifiers to interpret x and y as being elements of $\mathbf{N}$, and replace $\in$ with $<$. This produces the statement

$$\exists x \in \mathbf{N} \text{ so that } \forall y \in \mathbf{N}\,(\neg(y < x)).$$

This is a true statement; the number 0 is in $\mathbf{N}$, and for all other natural numbers $y \in \mathbf{N}$, y is not less than 0. Therefore, we conclude that $\mathbf{N} \vDash \varphi$.

❀**Definition 2.5.4:** Let Σ be a collection of sentences in the language of set theory. We say (M, ε) is a model of Σ when (M, ε) models every sentence in Σ: that is for all $\varphi \in \Sigma$, $M \vDash \varphi$.

In summary, we can define a collection of rules (sentences) that we would like to be true about our set theoretic “universe”; rules which might say things like "the empty set exists", or "if I have two sets A and B, then $\{A, B\}$ is also a set". We often call such sentences axioms.

A model would then be one particular universe in which all these rules (axioms) hold true. If axioms are thought of as abstract rules, then a model could be thought of as the axioms “put into practice”.

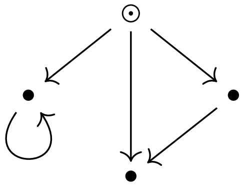
3 SETS AS GRAPHS

To discuss axioms systems which omit or contradict Foundation we will borrow much terminology from graph theory. We survey the relevant definitions below.

3.1 Pictures of Sets

We can represent sets with graphs in a very natural way. For our purposes, a graph G will consist of a collection of vertices (sometimes called nodes) denoted $V(G)$, and a binary relation $E(G) \subseteq V(G) \times V(G)$. If $(a, b) \in E$, we denote this $a \to b$, and say there exists an edge from vertex a to vertex b, or that b is a child of a. A path from a_0 to a_n is a *finite* sequence of edges $a_0 \to a_1 \to a_2 \cdots \to a_n$. Note that our edges are directed, meaning $a \to b$ does not imply $b \to a$, and that we allow loops, meaning it's possible to have an edge $a \to a$. Lastly, we do not allow double edges; there can be no two distinct edges from a to b in $E(G)$.

We'll focus our attention on accessible pointed graphs, which are graphs that have a distinguished node called the point of the graph and a path from the point to any other node in the graph.



☆Figure 3.1.1: An example of an apg. Following convention, the point is at the top of the diagram.

It is straightforward to depict many of the sets we know and love using apgs, where nodes represent sets and edges represent set membership. For example, the apg below can be thought to represent the von Neumann ordinal $2 = \{\varnothing, \{\varnothing\}\}$.

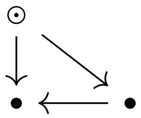


Figure 3.1.2: An apg depiction of the set $2 = \{\varnothing, \{\varnothing\}\}$

The node on the bottom left has no children, and so it is a natural candidate to represent the set with no elements (the empty set). The node on the bottom right has only one child, and we just decided that child represents the empty set, so therefore this node should represent the set $1 = \{\varnothing\}$. Thus, the point of the apg (and therefore the entire apg) can be thought to represent the set 2.

Note that a set could have many apgs that depict it. For example, the following apg could also be said to depict the von Neumann ordinal 2.

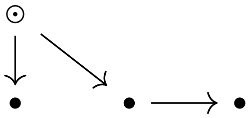


Figure 3.1.3: An alternative depiction of the set 2.

Let's introduce a definition which will allow us to make the notion of depiction more formal.

Definition 3.1.4: Given an apg G, a decoration of G is an assignment of elements of some set-theoretic universe to *each* node n in G so that the elements of the set assigned to n are precisely the sets assigned to the children of n. That is, a decoration assigns to each $n \in V(G)$ some set $d(n)$ so that $d(n) = \{d(n') \mid n \to n'\}$. We say an apg G is a picture of the set S if there exists a decoration which assigns S to the point of G.

We note that there is a notational "flip" that occurs when we use a decoration to turn an apg into a set, since by notational convention we want $a \to b$ to mean that *a contains b* as an element. That is if d is a decoration of apg G, then for all nodes $a, b \in V(G)$ we have

$$a \to b \quad \Leftrightarrow d(b) \in d(a).$$

The following definition from set theory will prove useful in our discussion of pictures.

❀**Definition 3.1.5:** Given a set X, the transitive closure of X, denoted $\mathrm{TC}(X)$, is defined as the smallest set transitive set which contains X as a subset.

One can think of $\mathrm{TC}(X)$ as being the set containing all the elements in X, along with all the elements of elements of X, along with all the elements of elements of elements of X, and so on. Following [9], $\mathrm{TC}(X)$ can also be defined recursively as follows.

$$T_0 = X$$

$$T_{n+1} = \cup T_0$$

$$\mathrm{TC}(X) = \cup_{n \in \omega} T_n$$

✧**Lemma 3.1.6:** If an apg G is a picture of the set X as witnessed by decoration d, then d is a surjection on $\mathrm{TC}(X)$.

Proof. Let $x \in \mathrm{TC}(X)$. Define recursively

$$T_0 = X$$

$$T_{n+1} = \cup T_n$$

By the above definition we know that if $x \in \mathrm{TC}(X)$ then $x \in T_n$ for some $n \in \omega$.

If $n = 0$ then $x \in T_0 = X$. Because G is a picture of X it follows that G must have point p and $d(p) = X$. Because d is a decoration, $d(p) = X = \{d(a) \mid p \to a\}$, and since $x \in X$ it follows that there is some $a \in V(G)$ so that $d(a) = x$, and our claim is proven.

For $n > 0$ we proceed by induction. Assume for all $m < n$ that $a \in T_m$ implies that a is in the image of d. Then $x \in T_n = \cup T_{n-1}$, and so $x \in y$ for some $y \in T_{n-1}$. By our induction hypothesis y is in the image of d, and so there is some node $a \in V(G)$ so that $d(a) = y = \{d(b) \mid a \to b\}$. Because $x \in y$ it then follows that $x = d(b)$ for some $b \in V(G)$. We then conclude that d is a surjection on $\mathrm{TC}(X)$. ■

✧**Theorem 3.1.7(Aczel, [1]):** Every set has a picture.

Proof. Let S be a set. We'll define an apg G as follows:

- $V(G) = \mathrm{TC}(S) \cup \{S\}$, that is the transitive closure of S along with the set S itself.
- $(a, b) \in E(G)$ if and only if $b \in a$.
- S is the point of G.

Then G is indeed an apg, and the identity function on G is a decoration of G assigning S to the point of G. Therefore, G is a picture of S. ■

3.2 Pictures of Sets in ZFC

We recall that Foundation implies there is no set with an infinite descending membership chain $x_0 \ni x_1 \ni x_2 \ni \ldots$. Sets which have no such infinite membership chain are called well-founded. Sets which *do* have such an infinite membership chain are called ill-founded. Thus Foundation asserts that no sets are ill-founded.

As a result, no apg with an infinite path $n_0 \to n_1 \to n_3 \to \ldots$ can be a picture of a ZFC set. Similarly, if an apg has no infinite path, we also say that it is well-founded.

✧**Theorem 3.2.1(Aczel, [1]):** Every well-founded apg has a unique decoration.

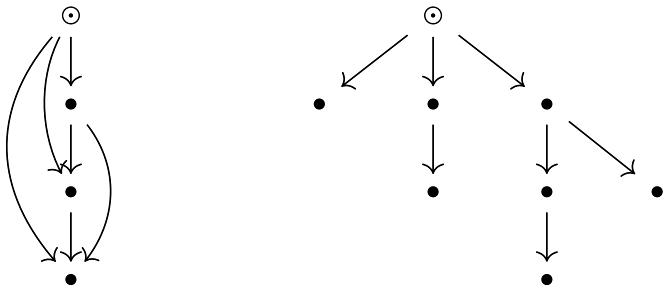
Proof. Let G be a well-founded apg with point p. Then any path starting at p in G terminates. For any node a that is n steps away from a terminal node (a node with no children), we assign $d(a) = \{d(a') \mid a \to a'\}$. Because each path through G is well-founded, $d(a)$ is defined for each $a \in V(G)$. Because decorations must respect set membership, this resulting decoration will be unique. ■

3.3 Introducing Exactness

We will introduce a property of decorations which will be crucial to the rest of our discussion.

Definition 3.3.1: If an apg G is a picture of a set X and the corresponding decoration is injective (meaning distinct nodes in $V(G)$ are assigned to distinct sets), then we say that G is an exact picture of X.

❏ **Example 3.3.2(Aczel, [1]):** The apg on the left below is an exact picture of the von Neumann ordinal $3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$. The apg on the right is a picture of 3 which is not exact.



☆**Figure 3.3.3:** Two apgs which are pictures of the von Neumann ordinal $3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. The apg on the right is not an exact picture; for example it contains 4 distinct nodes which will all be assigned the empty set.

In a sense, exact pictures of sets are the “best" pictures because they omit redundant nodes. Another advantage we'll see of considering exact pictures is that while a set may have many pictures, it will only have one exact picture (up to graph isomorphism).

❀**Definition 3.3.4:** Let $\mathrm{ZFC}^{-\mathrm{f}}$ denote the axiom system obtained by removing Foundation from ZFC.

✧**Theorem 3.3.5:** In $\mathrm{ZFC}^{-\mathrm{f}}$ every set has (up to isomorphism) a unique exact picture.

Proof. Let G, H be apgs that are both exact pictures of some set X. Then there exist injective decorations d_1 and d_2 from $V(G)$ and $V(H)$ respectively. Because both G and H are pictures of X, by Lemma 3.1.6 it follows that both d_1 and d_2 are surjections on $\mathrm{TC}(X)$.

Define a map $f : V(G) \to V(H)$ by $g \mapsto d_2^{-1}(d_1(g))$. That is, node g is sent to the node h in H so that $d_1(g) = d_2(h)$. Because d_1 and d_2 are both surjections onto the same set, it follows that $d_1(g)$ is in the image of d_2. Because d_2 is an injective function d_2^{-1} is also an injective function, and because d_1 is injective we know f is injective as well.

To see that f is a surjection, let $h \in V(H)$. Then h is assigned some set by d_2, which is some set in the transitive closure of X, which must then be the image by d_1 of some node g in $V(G)$. Thus $f(g) = h$.

Lastly, to see that f is a graph homomorphism, suppose there exists an edge $g_1 \rightarrow g_2$ in G. Then by decoration properties it must be the case that $d_1(g_1) \ni d_1(g_2)$, and again by decoration properties that there exists an edge $d_2^{-1}(d_1(g_1) \rightarrow d_2^{-1}(d_1(g_2))$. An identical arguments yields the converse; thus, f is a graph homomorphism and therefore an isomorphism. ■

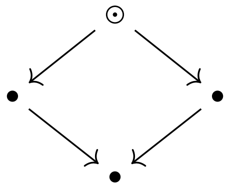
3.4 Exactness and Ill-Founded Sets

Oftentimes, the notions we've introduced thus far (e.g. apgs, decorations, pictures, exactness) are utilized to study ill-founded set theories, which are set theories which omit or explicitly violate Foundation and allow for the existence of ill-founded sets.

As discussed in [13] and will be demonstrated with Example 3.4.4, it turns out that $\mathrm{ZFC}^{-\mathrm{f}}$ is often "too vague" to be a desirable system. Thus set theorists wishing to study ill-founded theories will strengthen $\mathrm{ZFC}^{-\mathrm{f}}$ by including an additional axiom which contradicts Foundation and specifies which ill-founded sets exist. These supplemental axioms are often called anti-foundation axioms. Four anti-foundation axioms will be introduced in the following two chapters.

We've seen that each set has (up to isomorphism) one exact picture. Thus it is often convenient to phrase anti-foundation axioms as statements about which apgs are exact pictures. Because each set has a unique corresponding exact picture, if we know which apgs are exact, we will know which sets exist.

First, we'll note that some apgs can never be exact pictures, even by the most permissive anti-foundation axioms. Take the below apg as an example.



☆**Figure 3.4.1:** An apg which cannot be an exact picture.

Because this apg is well-founded, we know it has only one decoration; in fact, it is a picture of the set $\{\{\varnothing\}\}$. However, since both nodes directly beneath the point must both be assigned the set $\{\varnothing\}$, this decoration is not injective, and so the apg cannot be an exact picture.

❀**Definition 3.4.2:** We say an apg is extensional if and only if no two distinct nodes have the same children.

✧**Lemma 3.4.3:** A non-extensional apg can never be an exact picture.

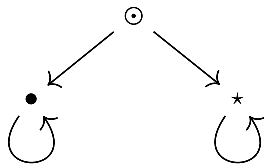
Proof. Suppose G is a non-extensional apg. Then there exist nodes $n, m \in V(G)$ which have the same set of children. Let d be a decoration of G. Then by definition

$$d(n) = \{d(n') \mid n \to n'\} = \{d(m') \mid m \to m'\} = d(m).$$

Therefore d is not injective. Since no decoration of G can be injective, we conclude that G cannot be an exact picture. ∎

For some apgs, however, the question of whether or not they are the exact picture of some set is more difficult to answer.

❏ **Example 3.4.4(Rieger, [13]):** Consider the following apg, which we'll call G^*.



☆**Figure 3.4.5:** An apg that may or may not be an exact picture.

In ZFC, G^* is not an exact picture, since it is ill-founded and thus cannot be a picture of any set at all. In ZFC$^{-\mathrm{f}}$, however, the situation is vague. G^* is an extensional apg since no two nodes have the same set of children, so perhaps it is possible for G^* to be exact in some ill-founded set theory. If this apg is the picture of some set S, then $S = \{\bullet, \star\}$ where $\bullet = \{\bullet\}$ and $\star = \{\star\}$. If it is to be an exact picture, then it must hold that $\bullet \neq \star$. But, is it true that $\bullet \neq \star$?

It turns out that ZFC$^{-\mathrm{f}}$ on its own cannot tell us the answer. Extensionality is our only tool for determining whether two sets are equal, and in this case it tells us that $\bullet = \star$ if and only if their elements are identical; that is, if and only if $\bullet = \star$. It is consistent with ZFC$^{-\mathrm{f}}$ to assume either that $\bullet = \star$ or $\bullet \neq \star$.

This example highlights that *if we want to know exactly which apgs are and are not exact pictures, we need to supplement ZFC^{-f} with some new axiom.* In particular the system ZFC$^{-\mathrm{f}}$ does not have a concrete notion of exactness the way that ZFC does, and so the property of "exactness" will be mutable in ZFC$^{-\mathrm{f}}$. Thus when we say an apg is "exact", we'll need to be precise about which strengthening of ZFC$^{-\mathrm{f}}$ it is exact in.

❀**Definition 3.4.6:** An apg is ZFC-exact if it is an exact picture of a set in ZFC.

Then, if our goal is to expand the universe of sets by allowing for non-well-founded sets, an axiom that tells us exactly which apgs are exact pictures and will also tell us exactly which non-well-founded sets exist (since each set has one exact picture).

4 THREE ANTI-FOUNDATION AXIOMS

The following chapter will introduce three anti-foundation axioms attributed to Peter Aczel and developed in [1], which are among the primary axioms considered in ill-founded set theory. A final anti-foundation axiom will be the subject of lengthier discussion in the following chapter.

Aczel proves in [1] that when any one of these axioms are added to $\mathrm{ZFC}^{-\mathrm{f}}$ the resulting system is consistent if and only if ZFC is consistent. Additionally, these four axioms are incompatible in the sense that no one can be proven from any of the others (assuming ZFC is consistent).

4.1 The Anti-Foundation Axiom (AFA)

We'll start with the anti-foundation axiom that has garnered the most attention. It is also the most restrictive, in that it allows for the "fewest" number of new ill-founded sets.

Confusingly, this axiom is called *the* Anti-Foundation Axiom, often abbreviated AFA. In this text, the phrase "anti-foundation axiom" (in lowercase) will refer to any axiom which contradicts Foundation. When referring to *the* Anti-Foundation Axiom, its name will be capitalized or we will use the shorthand AFA.

✧**The Anti Foundation Axiom (AFA):** Every apg has a *unique* decoration.

Many texts point out that AFA is equivalent to the conjunction of the following two statements:

- AFA_1: Every apg has at least one decoration.
- AFA_2: Every apg has at most one decoration.

AFA_1 is the permissive component of AFA, and AFA_2 is the restrictive component. AFA_1 expands our collection of “acceptable" apgs; whereas in ZFC only well-founded apgs could be pictures of sets, now any apg can be a picture of a set. On the other hand, AFA_2 asserts that many of these newly-acceptable apgs are merely alternate representations of the same set.

A few examples will help shed some light on the new sets we can consider by adopting AFA.

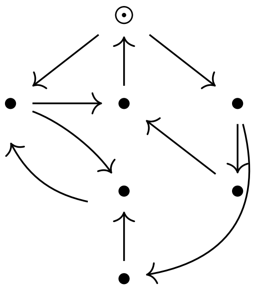
❀**Definition 4.1.1:** Let ZFA refer to the axiom system containing the axioms of ZFC^{-f} as well as AFA.

❏ **Example 4.1.2:** By AFA, the following apg is the picture of a set in ZFA.

☆**Figure 4.1.3:** An apg depicting an ill-founded set.

The set corresponding to this apg is the set whose only element is itself. This set is often denoted as $\Omega = \{\Omega\} = \{\{\Omega\}\} = \ldots$ and has the infinite-descending membership chain $\Omega \in \Omega \in \Omega \in \ldots$, thus violating Foundation.

❏ **Example 4.1.4:** Similarly, by AFA the following apg is also the picture of some set in ZFA.



☆**Figure 4.1.5:** A slightly convoluted ill-founded apg.

But which set is it? First, we note that every node in this apg has a child. Therefore, the decoration which assigns each node to the set Ω would indeed be a valid decoration. Since AFA claims each apg has *only one* decoration, it must then be that this apg is a (non-exact) picture of Ω.

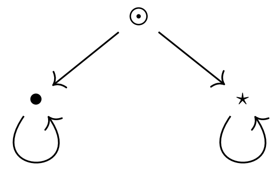
✧**Theorem 4.1.6(Aczel, [1]):** In ZFA an apg is a picture of Ω if and only if every node of the apg has a child.

Proof. Assume we have an apg with root a and decoration d so that $d(a) = \Omega$. Then if b is any node of the apg there must be a path $a = a_0 \to a_1 \to .. \to a_n = b$ so that $d(b) = d(a_n) \in d(a_{n-1}) \in ... \in d(a_0) = \Omega$. Since Ω is the only element of Ω, it follows that $d(b) = \Omega$. Since Ω has an element, it follows that b must have a child. Thus every node in the apg must have a child.

On the other hand, suppose we have an apg where every node has a child. Then assigning Ω to each node of the apg provides a decoration of the apg, and by AFA this decoration must be unique. Therefore the apg is a picture of Ω. ■

This theorem highlights the restrictiveness of AFA; any apg in which every node has a child, no matter how vast or complex this network of edge relations is, is equivalent to the apg with a single vertex and an edge to itself.

We note then that AFA answers our question from Example 3.4.4 of “Is G^* an exact picture?" with a definitive “no”. Since every node in G^* has a child, G^* can only be a picture of Ω. The decoration showing that G^* is a picture of Ω assigns Ω to each node in the diagram, and thus is not an injective decoration. Therefore G^* is not an exact picture in AFA.



☆**Figure 4.1.7:** An apg which is *not* an exact picture in ZFA.

4.2 An Alternative Formulation of AFA

We stated previously that a potentially advantageous strategy for expressing anti-foundation axioms is by writing them as statements about which apgs are and are not exact pictures. This strategy is often the most straightforward way to state the three anti-foundation axioms we have yet to introduce. When introducing AFA, however, this approach tends to be less intuitive, and so the formulation of AFA given in the previous subchapter is often the preferred one.

Nevertheless, it is possible to formulate AFA as an expression about exact pictures, which we will venture to do now. First, we'll need to define the notion of bisimulation.

Definition 4.2.1: Let A, B be apgs with points a' and b' respectively. A relation $\sim$ between the vertices of A and B (that is $\sim \subseteq V(A) \times V(B)$) is a bisimulation if for all $a_1, a_2 \in V(A)$ and $b_1, b_2 \in V(B)$ the following conditions hold:

1. If $a_1 \sim b_1$ and $a_1 \to a_2$ then there is a b_* so that $b_1 \to b_*$ and $a_2 \sim b_*$.
2. If $a_1 \sim b_1$ and $b_1 \to b_2$ there is an a_* so that $a_1 \to a_*$ and $a_* \sim b_2$
3. $a' \sim b'$.

Definition 4.2.2: If a bisimulation exists between A and B, we say they are bisimilar, which we write as $A \cong_{\mathrm{B}} B$.

✧**Theorem 4.2.3:** $\cong_B$ defines an equivalence relation between apgs.

Proof. We note that for any apg G the identity relation $\sim_{\text{id}}$ on the nodes of G defined by

$$a \sim_{\text{id}} b \Leftrightarrow a = b$$

is certainly a bisimulation. Thus $\cong_B$ is reflexive.

If $G_1 \cong_B G_2$ then there exists a bisimulation $\sim \subseteq V(G_1) \times V(G_2)$. Then the converse relation $\text{conv}(\sim) \subseteq V(G_2) \times V(G_1)$ will also be a bisimulation due to the symmetry of conditions 1 and 2 from the definition of bisimulation. Thus $\cong_B$ satisfies symmetry.

For transitivity, suppose we have apgs G_1, G_2, and G_3 with points g_1, g_2, and g_3 respectively and $G_1 \cong_B G_2 \cong_B G_3$. Then there exist bisimulations $\sim_\alpha$ between G_1 and G_2 and $\sim_\beta$ between G_2 and G_3. Define a relation $\sim \subseteq V(G_1) \times V(G_3)$ by

$$a \sim b \Leftrightarrow \exists c \in V(G_2) \text{ with } a \sim_\alpha c \text{ and } c \sim_\beta b.$$

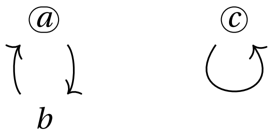
Let's verify that $\sim$ is indeed a bisimulation. Let $a_1, a_2 \in G_1$, $b_1 \in G_3$, $a \sim b$ and $a_1 \rightarrow a_2$. To satisfy condition 1, we want to show that there exists $b_2 \in V(G_3)$ so that $b_1 \rightarrow b_2$ and $a_2 \sim b_2$.

By definition of $\sim$ there then exists $c_1 \in G_2$ with $a_1 \sim_\alpha c_1$ and $c_1 \sim_\beta b_1$. Additionally by definition of bisimulation there must exist $c_2 \in V(G)$ so that $c_1 \rightarrow c_2$ as well as $a_2 \sim_\alpha c_2$. Again then by definition of bisimulation there must exist $b_2 \in V(G_3)$ so that $b_1 \rightarrow b_2$ and $c_2 \sim_\beta b_2$. Since $a_2 \sim_\alpha c_2$ and $c_2 \sim_\beta b_2$, it follows that $a_2 \sim b_2$. We already know that $b_1 \rightarrow b_2$ holds. Thus, we have shown that condition 1 of bisimulation is satisfied.

The proof of condition 2 follows from an argument identical in structure to the one above. Lastly by definition of bisimulation it must be that $g_1 \sim_\alpha g_2$ and $g_2 \sim_\beta g_3$, and so $g_1 \sim g_3$ and condition 3 of bisimulation holds.

We thus conclude that $\sim$ is a bisimulation. Therefore $G_1 \cong_{\mathrm{B}} G_3$, and so $\cong_{\mathrm{B}}$ is indeed transitive and therefore an equivalence relation. ■

❑ **Example 4.2.4:** The two apgs depicted below are bisimilar. Note that a is the point of the apg on the left.



☆**Figure 4.2.5:** Two bisimilar apgs.

To see this, we'll define the relation $\sim$ on the vertices of the two apgs as follows:

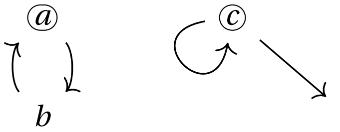
$$\sim := \{(a,c),(b,c)\}$$

and show that $\sim$ is a bisimulation.

For condition 1 above, $a \sim c$ and $a \to b$. We note then that $c \to c$ and $b \sim c$. An identical argument for $b \sim c$ shows that condition 1 is satisfied.

For condition 2, $a \sim c$ and $c \to c$. We note that $b \to a$ and $b \sim c$, so condition 2 is satisfied. Lastly, $a \sim c$ so condition 3 is satisfied. Thus $\sim$ is a bisimulation between the two apgs, and they are bisimilar.

❑ **Example 4.2.6:** The two apgs depicted below are not bisimilar.



☆**Figure 4.2.7:** Two apgs that are not bisimilar.

We note that any relation $\sim$ that is to be a bisimulation between these two apgs, condition 3 demands that $a \sim c$. As a consequence of condition 1 it must hold that $b \sim d$. However, $b \sim d$ and $b \to a$ but d has no children, which violates condition 1. We thus conclude that no bisimulation can exist between the two apgs, so they are not bisimilar.

Note that some authors (such as [8]) define the notion of bisimilarity using the *largest bisimulation* between A and B, which is given by the union of all bisimulation between A and B. If $A \sim B$ for any bisimulation $\sim$, then they are also related by the largest simulation between A and B, and so these two notions are equivalent.

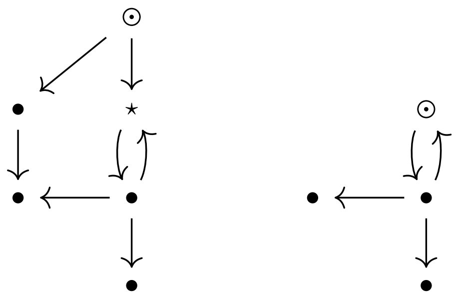
Our alternative formulation of AFA will also make use of the notion of descendant subgraph, which we'll define below.

Definition 4.2.8: Let G be an apg and $a \in V(G)$ be a node of G. Then $G[a]$ will denote the subgraph "below" a, that is the subgraph of G induced by the vertex set

$$\{b \in V(G) \mid \exists \text{ a finite path from } a \text{ to } b\}$$

that has point a. Such a subgraph is called a descendant subgraph.

❏ **Example 4.2.9:** The figure on the left depicts an apg G, and the figure on the right depicts the descendant subgraph $G[\star]$.



☆**Figure 4.2.10:** An apg and a descendant subgraph.

❀**Definition 4.2.11** Given an apg A, We then say that A is strongly extensional when there are no two distinct nodes $a_1, a_2 \in V(A)$ so that $A[a_1] \cong_{\mathrm{B}} A[a_2]$. That is, no two distinct nodes in A give rise to two bisimilar descendant subgraphs.

We're now ready to write the axiom AFA as a statement about exact pictures.

✧**Theorem 4.2.12(Aczel, [1]):** An apg is exact in ZFA if and only if it is strongly extensional. For proof see [1] page 28.

More details on bisimulation and this equivalence can be found in [17]. The origins of the idea of bisimulation are in state transition systems from computer science. Incurvati has stated in [8] that a graph being strongly extensional is roughly equivalent to the statement that it has "no distinct nodes such that exactly the same movements are possible along the edges departing from these nodes".

Very informally, if two apgs are bisimilar then they are equivalent in this very weak sense (much weaker than isomorphism, for example) involving possible movements along their nodes. If two distinct nodes in an apg have bisimilar descendant subgraphs, then from AFA's perspective these two descendant subgraphs are equivalent. Therefore to depict these descendant subgraphs as being attached to two different nodes is to redundantly depict two “equivalent" structures, violating the “injective nature" of exact pictures. This notion implies that bisimilar apgs must depict the same set.

However, if we replace bisimular with a stronger notion of what “equivalent" subgraph means, we can produce new anti-foundation axioms. Let’s move onto the next one!

4.3 Scott’s Anti-Foundation Axiom (SAFA)

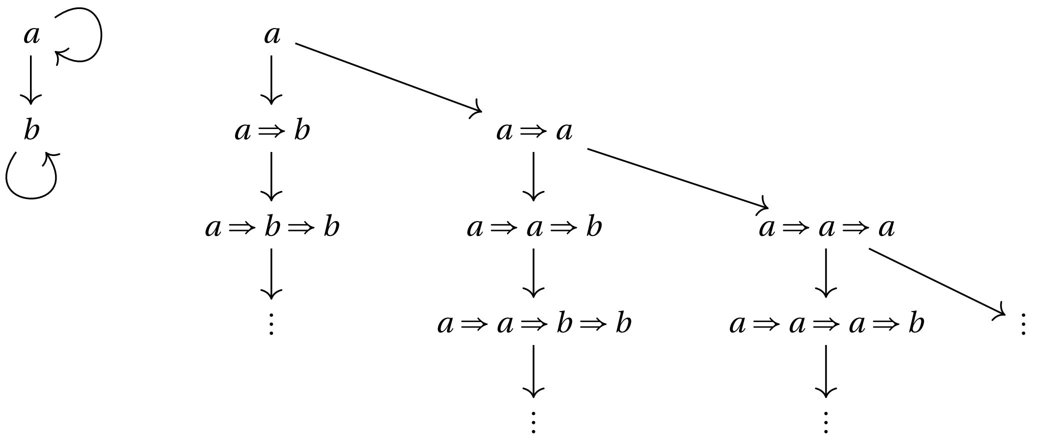
Our next anti-foundation axiom, though developed by Aczel in [1], first appeared in an (unpublished) talk by Dana Scott at the Stanford Congress of Logic, Methodology, and Philosophy of Science in 1960. This subchapter will borrow examples and exposition from [8].

❀**Definition 4.3.1:** Given an apg G with point n_0, the unfolding of G is the directed tree whose nodes are finite paths in G starting at n_0 (note that include the path consisting only of n_0 as a path of length zero). Edges in the unfolding are pairs of paths of the form.

$$(n_0 \rightarrow \ldots \rightarrow n, \quad n_o \rightarrow \ldots \rightarrow n \rightarrow n').$$

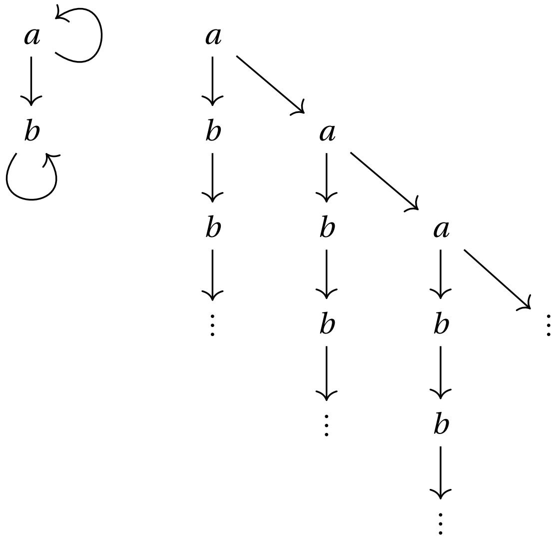
In other words, if G is an apg and T is its unfolding, then the nodes of T are all the paths along G starting from the point, and there is an edge $t_1 \rightarrow t_2$ in T if the path t_2 extends the path t_1 by a single step.

❏ **Example 4.3.2:** The following figure depicts an apg with point a and its unfolding.



☆**Figure 4.3.3:** An apg on the left and its unfolding on the right. Single-line arrows represent edges in the unfolding, and double-line arrows are used to denote paths in the apg.

This depiction of the unfolding is a bit unwieldy; we often condense notation and simply denote each node in the unfolding with the last node of the path it represents.



☆**Figure 4.3.4:** The same apg and unfolding from the previous example with condensed notation. Note that in the unfolding, for example, the right-most node labeled "a" actually represents the path $a \to a \to a$ in G.

Similar to our approach in defining AFA in the previous subchapter, we will now define a property of apgs which is based on distinct nodes having "distinct enough" subgraphs.

Definition 4.3.5: We say an apg G with point n_0 is Scott-extensional when there are no distinct nodes n, n' with corresponding paths $n_0 \to \ldots \to n$ and $n_0 \to \ldots \to n'$ in G so that the subtree below $n_0 \to \ldots \to n$ in the unfolding of G is isomorphic to the subtree below $n_0 \to \ldots \to n'$.

Looking to our apg and its unfolding in Figure 4.3.4, it is clear that there is no path ending in a that has a subtree isomorphic to a path ending in b (since subtrees of paths ending in b are always path graphs, and subtrees of paths ending in a are never). Thus, the apg in Figure 4.3.4 is Scott-extensional.

✧**Scott's Anti-Foundation Axiom (SAFA):** An apg is an exact picture if and only if it is Scott-extensional.

It's interesting to note that SAFA is a more permissive axiom than AFA. To give one example, the apg in Figure 4.3.4 is Scott-extensional, and is thus a picture of the set $X \neq \Omega$ with $X = \{X, \Omega\}$. However by AFA the apg is merely a non-exact picture of Ω (as can be verified by the fact that the apg is not strongly extensional, and also that every node in the apg has a child).

✧**Theorem 4.3.6(Aczel, [1]):** If an apg is strongly extensional, then it is Scott-extensional.

Proof. Suppose by way of contrapositive that an apg G is not Scott-extensional. Then there exists nodes $a, b \in V(G)$ and an isomorphism f between the unfoldings below a and b.

We'll construct a bisimulation $\sim$ on the nodes of the descendant subgraphs $G[a]$ and $G[b]$ by $v \sim w$ if and only if there exists a path ending in v which is mapped by f to a path ending in w in the unfolding of G. We'll verify that this relation satisfies the conditions of a bisimulation.

First, suppose we're given $a_1 \in G[a]$ and $b_1 \in G[b]$ so that $a_1 \sim b_1$. Then there exist a path to a_1 so that its subtree in the unfolding is isomorphic to the subtree below some path to b_1. Let $a_2 \in G[a]$ with $a_1 \to a_2$. Then by the existence of isomorphism f there must be a corresponding $b_2 \in G[b]$ with $b_1 \to b_2$. By definition of f, it must be then that $a_2 \sim b_2$, since the path to a_1 can be extended to include a_2, whose subtree in the unfolding must then must be isomorphic to that of the path to b_1 extended by b_2. An identical argument establishes the symmetrical condition.

Lastly, because the subtree below a path to a is isomorphic to that below a path to b, f will map a to b, and so $a \sim b$. We have thus shown that $\sim$ is a bisimulation between the descendant subgraphs of two distinct nodes in G, and thus conclude that G is not strongly extensional. ■

Note that Scott-extensionality does not imply strong extensionality, as the apg in Figure 4.3.4 demonstrates.

4.4 Finsler's Anti-Foundation Axiom (FAFA)

Our next axiom was formalized by Aczel in [1] based on a notion first attributed to Paul Finsler. We will continue the trend of defining an equivalence between apgs and then using this equivalence to choose our exact pictures. The equivalence will use this time is the notion of graph isomorphism.

Definition 4.4.1: We say apgs A, B with points a, b respectively are isomorphic when there exists a bijection $f : V(A) \to V(B)$ so that $f(a) = b$ and that for any $a_1, a_2 \in V(A)$ we have $a_1 \to a_2$ if and only if $f(a_1) \to f(a_2)$.

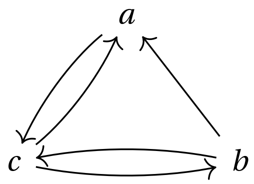
This definition is identical to the usual notion of graph isomorphism, save for the fact that we require the point of one apg to be mapped to the point of the other.

❀**Definition 4.4.2:** We say an apg G is Finsler-extensional (also occasionally called isomorphism-extensional) when there are never distinct nodes $n, n' \in V(G)$ so that $G[n]$ (the descendant subgraph below n) is isomorphic to $G[n']$ (the descendant subgraph below n').

✧**Finsler's Anti-Foundation Axiom (FAFA):** An apg is an exact picture of a set if and only if it is Finsler-extensional.

We'll prove shortly that Finsler-extensionality is a weaker condition than Scott-extensionality, and thus that FAFA is a more permissive axiom than SAFA and provides for the existence of even more ill-founded sets. To illustrate this fact, let's seen an example of an apg which is Finsler-extensional but not Scott-extensional.

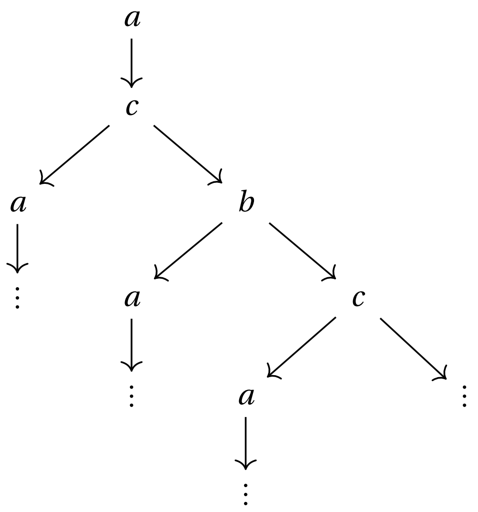
❏ **Example 4.4.3: (Aczel, [1])** Consider the following apg we'll call G. Note the point of G is a.



☆**Figure 4.4.4:** An Finsler-extensional apg.

We note that because every node in G is accessible along a finite path starting from any other node in G, we have that $G[a], G[b]$, and $G[c]$ all have the name set of nodes, but different points. Because there is no automorphism on G besides the trivial automorphism, we conclude that G is Finsler-extensional and therefore an exact picture by FAFA. In particular, it is an exact picture of the set $A = \{B, C\}$ where $B = \{A\}$, $C = \{A, B\}$ and $A \neq B \neq C$

To see that G is not Scott-extensional, let's take a look at its unfolding.



☆**Figure 4.4.5:** The unfolding of the apg G from the previous example.

We note that there are many paths in G which end in distinct nodes that give rise to isomorphic subtrees in the unfolding. For example, the subtree below the path $a \to c$ is isomorphic to the subtree below the path $a \to c \to b$. We conclude then that G is not Scott-extensional, and is therefore not an exact picture according to SAFA.

✧**Theorem 4.4.6: (Aczel, [1]):** If a graph is Scott-extensional, then it is Finsler-extensional.

Proof. Let G be an apg with point p and assume by way of contrapositive that G is not Finsler-extensional. Then there exist nodes $a, b \in V(G)$ so that $G[a] \cong G[b]$. Because G is accessible there then exist paths $p = a_0 \to a_1 \to \ldots \to a_n = a$ and $p = b_0 \to b_1 \to \ldots \to b_m = b$.

Consider then the unfolding of G, which we denote G'. If we look at the subtrees of G' below the paths $p = a_0 \to a_1 \to \ldots \to a_n = a$ and $p = b_0 \to b_1 \to \ldots \to b_m = b$, these two subtrees must be isomorphic because the subgraphs below a and b are isomorphic. Thus we conclude that G is not Scott-extensional. ■

5 ONE MORE ANTI-FOUNDATION AXIOM

Our last anti-foundation axiom is again developed in its presented form by Aczel in [1] but is originally credited to Maurice Boffa in [3]. In a sense, it allows for the existence of as many ill-founded sets as possible without violating the Axiom of Extensionality.

5.1 Boffa's Weak Axiom

Boffa's Anti-Foundation Axiom is the conjunction of two statements, which are often denoted BA_1 (also called "Boffa's Weak Axiom") and BA_2. We are already well-equipped to understand BA_1, while BA_2 is a subtler axiom which will merit further discussion.

✧**Axiom (BA_1):** An apg is an exact picture if and only if it is extensional.

Recall from subchapter 3.4 that if an apg is not extensional, then no decoration of it can be exact in any strengthening of $\mathrm{ZFC}^{-\mathrm{f}}$. Thus extensionality is the minimum requirement for a picture to be exact, and BAFA asserts that all apgs meeting this minimum threshold are in fact exact.

✧**Theorem 5.1.1 (Aczel, [1]):** If an apg is Finsler-extensional, then it is extensional.

Proof. We proceed by contrapositive and assume that an apg G is not extensional. Then there exists nodes $a, b \in V(G)$ that have the same set of children. Then $G[a]$ will be isomorphic to $G[b]$, and so G is not Finsler-extensional. ■

AFA, SAFA, and FAFA (as well as standard ZFC) all respect a principle referred to in [16] as isomorphism-extensionality (IE), which states that sets with isomorphic membership structure are equal (since isomorphic apg structure is the threshold for equality and FAFA, and both SAFA and AFA require stricter conditions for equality). However, BAFA does not respect IE.

An illuminating example of how these theories interact with isomorphism extensionality is in their quantities of Quine atoms.

✿**Definition 5.1.2:** A set x is a Quine atom (also called a reflexive set) when $x = \{x\}$.

As quick note on the utility of Quine atoms (discussed in [16]), if one wishes to add atoms (objects which have no elements but that are distinct from the empty set) to ZFC, the typical approach requires altering the Axiom of Extensionality. An alternative approach is to assert the existence of the desired number of Quine atoms, which produces a set theory in which Extensionality is preserved at the cost of Foundation.

✧**Theorem 5.1.3(Aczel, [1]):** FAFA implies there exists exactly one Quine atom.

Proof. We begin by noting that the apg consisting of a single self-connected node is Finsler-extensional. Thus FAFA implies a Quine atom exists.

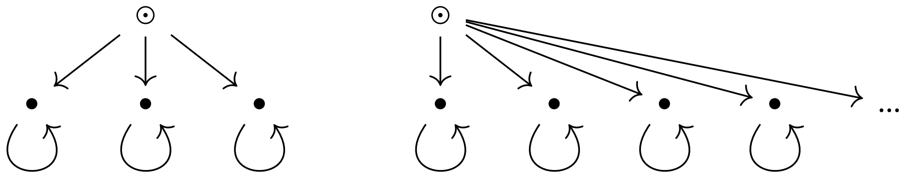
For uniqueness, assume that x, y are Quine atoms. Then the apgs below are an exact picture of x and y respectively, since they have injective decorations d and d' with $d(\bullet) = x$ and $d(\star) = y$.

Since these apgs are isomorphic, by FAFA (and IE) they must depict the same set. Therefore $x = y$, and we conclude that FAFA implies the existence of exactly one Quine atom. ■

Because FAFA implies there is only one Quine atom and is a strictly more permissive axiom than both AFA and SAFA, it follows that AFA and SAFA allow for the existence of at most one Quine atom (it is easy to verify that a Quine atom exists by both axioms). However BAFA turns out to be far more generous.

✧Theorem 5.1.4(Aczel, [1]): Let A be a set, and let $|A|$ denote the cardinality of A. Then BA_1 implies there exist $|A|$-many Quine atoms. That is, each element of A can be assigned a unique Quine atom.

Proof. Consider the apg G consisting of a point with $|A|$-many children, each of which has itself as its only child.



✩Figure 5.1.5: Examples of G when $|A| = 3$ and $|A| = \omega$ respectively.

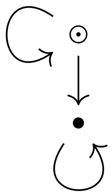
Then G is extensional, and so it is an exact picture. Thus the corresponding decoration of G is injective, meaning each child of the point is assigned a distinct set. Thus G provides for the existence of $|A|$-many Quine atoms. ■

We see then that BAFA set theory has the unique property that it asserts the existence of many sets which all have isomorphic membership structure. This fact will motivate the definition of BA_2, which we'll discuss next.

5.2 BA$_2$

At first glance one might be puzzled why BA$_1$ is insufficient to define a satisfactory anti-foundation axiom on its own, and why an additional axiom is necessary. The following example may prove helpful.

❑ **Example 5.2.1:** We've already seen that BA$_1$ implies the existence of arbitrarily many distinct Quine atoms. Let Q be one such Quine atom. Suppose we then want to consider the set $A := \{A, Q\}$. Does such a set exist? Indeed this set would have the structure of the following apg.



☆**Figure 5.2.2:** An apg with the same structure as our desired set A.

One can quickly verify that this apg is extensional, and so by BA$_1$ it must be an exact picture of some set, which we'll call A'. This means an injective decoration exists assigning A' to the node of our apg. However, because there are now many sets isomorphic to Q, there is no guarantee that the bottom node of our apg must be assigned to Q by our injective decoration (in fact there are class-many other sets it could be assigned). Therefore we cannot conclude that $A = A'$. BA$_1$ asserts the existence of a set isomorphic to A, but not the existence of A itself.

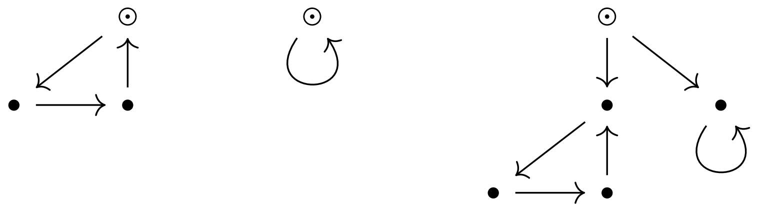
Constructing A by some other means (Pair, Union, etc) will prove difficult. For example, to use Pair to collect A and Q into the set $\{A, Q\}$ we would first have to know that the set A exists! Thus we need an additional axiom that will allow us to *extend* Q to create the set A. We'll follow the formulation of BA_2 presented in [10].

✧**Axiom (BA_2):** Every injective decoration of a descendant subgraph of an exact picture can be extended to an injective decoration of the whole graph.

Let's use the power of BA_2 to construct A. There is indeed an injective decoration assigning Q to the descendant subgraph of the apg consisting of only the bottom node. Thus, this decoration can be extended to an injective decoration on the whole graph which also assigns Q to this node, thus producing the desired set A.

In a similar vein to the previous example, a model of BA_1 might fail to satisfy even the most basic $\mathrm{ZFC}^{-\mathrm{f}}$ axioms, such as Pair.

❏ **Example 5.2.3:** Let G_1 and G_2 be two extensional apgs which are pictures of sets x_1 and x_2 in some model of BA_1. By BA_1 it must follow that these pictures are exact. We can then construct an extensional apg which is the "pair apg" of G_1 and G_2 which is the union of G_1 and G_2 with an additional node p above the node of G_1 and G_2 and point p.



☆**Figure 5.2.4:** From left to right: an extensional apg G_1, another extensional apg G_2, and the pair apg of G_1 and G_2.

Certainly the pair apg is extensional, so it is an exact picture of some set y. However, by the same argument as used in Example 5.1.4, there are class-many sets which are isomorphic to either x_1 or x_2. Thus there's no guarantee that $y = \{x_1, x_2\}$. However, by leveraging BA_2, we can *extend* the decorations on G_1 and G_2 to the pair apg, and thus ensure that we obtain a set $y = \{x_1, x_2\}$

As our examples demonstrate, the reason BA_2 is desirable is because BA_1 asserts the existence of numerous isomorphic yet distinct sets, and so to extend or make new sets from old, we need an axiom that allows us to "pick out" the set with the correct structure *and* the correct constituent elements.

✧ **Boffa's Anti-Foundation Axiom (BAFA)** is equivalent to the conjunction of BA_1 and BA_2.

As a note about our chosen presentation of BAFA, Aczel's original formulation of BAFA in [1] is the statement "Every exact decoration of a transitive subgraph of an extensional graph can be extended to an exact decoration of the whole graph". This statement is nearly identical to our BA_2, and in fact implies BA_1. To see this, for any apg G consider the empty subgraph of G. This subgraph is certainly transitive, and thus can be extended to an exact decoration of G. Thus every extensional graph is an exact picture. As we've seen previously, Extensionality demands the implication that every exact picture is an extensional graph. We've chosen to present BAFA as the conjunction $(\mathrm{BA}_1 \wedge \mathrm{BA}_2)$ for clarity and consistency with our presentation of the other anti-foundation axioms.

5.3 An Alternative Formulation of BAFA

Like the other anti-foundation axioms we've discussed so far, there are many alternative ways to present the axiom BAFA. The following form of BAFA as stated in [2] might feel more set-theoretic in its presentation. First, we'll list some relevant definitions.

Recall a set A is transitive when $x \in y \in A$ implies $x \in A$. Additionally, if A and A' are sets equiped with binary relations R and R' respectively, we say (A, R) is a transitive in (A', R') when $A \subseteq A'$ and for all $a \in A$ we have $\{x \in A \mid xRa\} = \{x \in A' \mid xR'a\}$.

✧**Axiom (BA$_3$):** Suppose (A, R) is transitive in (A', R') with R' extensional, that B is transitive set, and $f : (A, R) \to (B, \in)$ is an isomorphism. Then there exists a transitive set B' and function f' so that $B \subseteq B'$, $f \subseteq f'$, and $f' : (A', R') \to (B', \in)$ is an isomorphism.

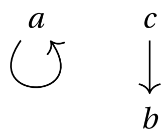
We will see shortly that BA$_3$ is equivalent to the conjunction of BA$_1$ and BA$_2$. However, we note that BA$_1$ and BA$_2$ are statements about extensional *apgs*, while BA$_3$ is a statement about extensional *relations*.

We can naturally consider an extensional relation (A, R) as representing an extensional graph G by taking $V(G)$ to be A and $E(G)$ to be the converse of R (we take the converse here since we typically want an edge $a \to b$ to mean $b \in a$). However, the resultant *graph* G need not be an *apg*. In particular G may fail to have a natural choice of point, and may fail to be accessible.

❏ **Example 5.3.1:** Consider the set A with extensional relation R below, which fails to be an apg when translated into a graph.

$$A = \{a, b, c\}$$

$$R = \{(a, a), (b, c)\}$$



✩**Figure 5.3.2:** A set A with an extensional relation R on the left, and the graph representation of (A, R) on the right. Note that there is no choice of point in the graph which will produce an *accessible* pointed graph.

However, given any extensional relation R on a set A, we will be able to construct an apg G

which extends (A, R) by adding new nodes and edges *above* the elements of A, but not below. When we consider the the descendant subgraph of G below (potentially multiple) elements of A, we will recover a graph isomorphic to (A, R). The following lemma outlines this process.

✧ **Lemma 5.3.3(Apg-ification Lemma):** Given any extensional relation R on a set A, we can construct an extensional apg G with point p and with injective $f : A \to G$ so that the descendant subgraph below all nodes in the image of f is isomorphic to (A, R).

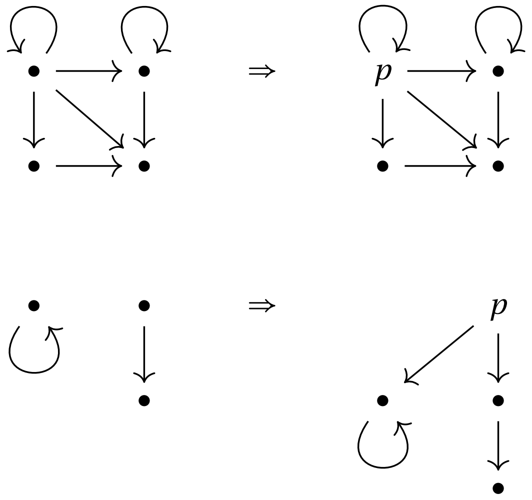
Proof. We consider (A, R) as the graph G where $V(G) = A$ and $E(G)$ is the converse of R (that is, $(a, b) \in E(G)$ if and only if $(b, a) \in R)$.

If G has a node p so that every node in $V(G)$ is a child of p (including p itself), then there is a unique such p by the extensionality of (A, R). In this case, we'll choose p to be the point of G; then G is surely pointed and accessible.

If no such p exists already in G, then we add a new node p to G and define $\neg(p \to p)$ as well as $p \to a$ and $\neg(a \to p)$ for all $a \in V(G)$ with $a \neq p$. In other words, p is a new node which has every other node as a child and which itself is the child of no node. Then G will still be extensional, and we may select p to be the point of G and turn G into an apg.

Then certainly A injects into G, and the graph consisting of the union of all descendant subgraph below all nodes in the image of this embedding will be isomorphic to (A, R). ■

❏ **Example 5.3.4:** Two examples of applications of the Apg-ification Lemma are presented below.



✩**Figure 5.3.5:** Two graphs representing sets with extensional relations on the left, and their respective apg-ifications on the right.

Now that we have the power to apg-ify extensional relations, we are ready to proceed with the proof of the desired equivalence.

✧**Theorem 5.3.6:** $(\mathrm{BA}_1 \wedge \mathrm{BA}_2)$ is equivalent to BA_3.

Proof. $(\Leftarrow)$ Assume BA_3, and let G be an extensional apg. Then the pair $(V(G), \leftarrow)$ where $\leftarrow$ is the conversed edge relation is a set with an extensional relation. Trivially $(\varnothing, \varnothing)$ is a set with a relation that is transitive in $(V(G), \leftarrow)$. Additionally there exists an isomorphism $f : (\varnothing, \varnothing) \to (\varnothing, \in)$.

Then by BA_3 there exists a transitive set B' with isomorphism f' from $(V(G), \leftarrow)$ to $(B', \in)$. Since f' is an isomorphism, it is equivalent to an injective decoration assigning nodes of G to elements of B'. Then G will be an exact picture of the set which is assigned to the node of G, and thus $\mathrm{BA}_3 \Rightarrow \mathrm{BA}_1$.

To see that $\text{BA}_3 \Rightarrow \text{BA}_2$, suppose that an apg G^* is an exact picture and that G is a descendant subgraph of G^* which has an injective decoration d assigning the set S to the node of G. We want to show that d can be extended to an injective decoration on all of G^*.

Since G^* is an exact picture of some set, it must be that $(V(G^*), \leftarrow)$ is extensional. Additionally by the definition of descendant subgraph we know that $(V(G), \leftarrow)$ is transitive in $(V(G^*), \leftarrow)$. Also, since d is an injective decoration on G, it must be that $(V(G), \leftarrow) \cong (\text{TC}(S) \cup \{S\}, \in)$ witnessed by isomorphism d.

By BA_3 there exists a transitive set B' with $\text{TC}(S) \cup \{S\} \subseteq B'$ and function f' which is an extension of d and an isomorphism between $(V(G^*), \leftarrow)$ and $(B', \in)$. As before, this isomorphism represents an injective decoration assigning nodes of $V(G^*)$ to elements of B'. Because $d \subseteq f'$, we have succeeded in extending the injective decoration d of the descendant subgraph to an injective decoration on the entire graph. We thus conclude that BA_2 is satisfied, so $\text{BA}_3 \Rightarrow (\text{BA}_1 \wedge \text{BA}_2)$.

($\Rightarrow$) Next we'll assume BA_1 and BA_2. Suppose that (A, R) is transitive in extensional (A', R') and that B is a transitive set with isomorphism $f : (A, R) \to (B, \in)$. We want to find a transitive set B' and function f' so that $B \subseteq B'$, $f \subseteq f'$, and $f' : (A', R') \to (B', \in)$.

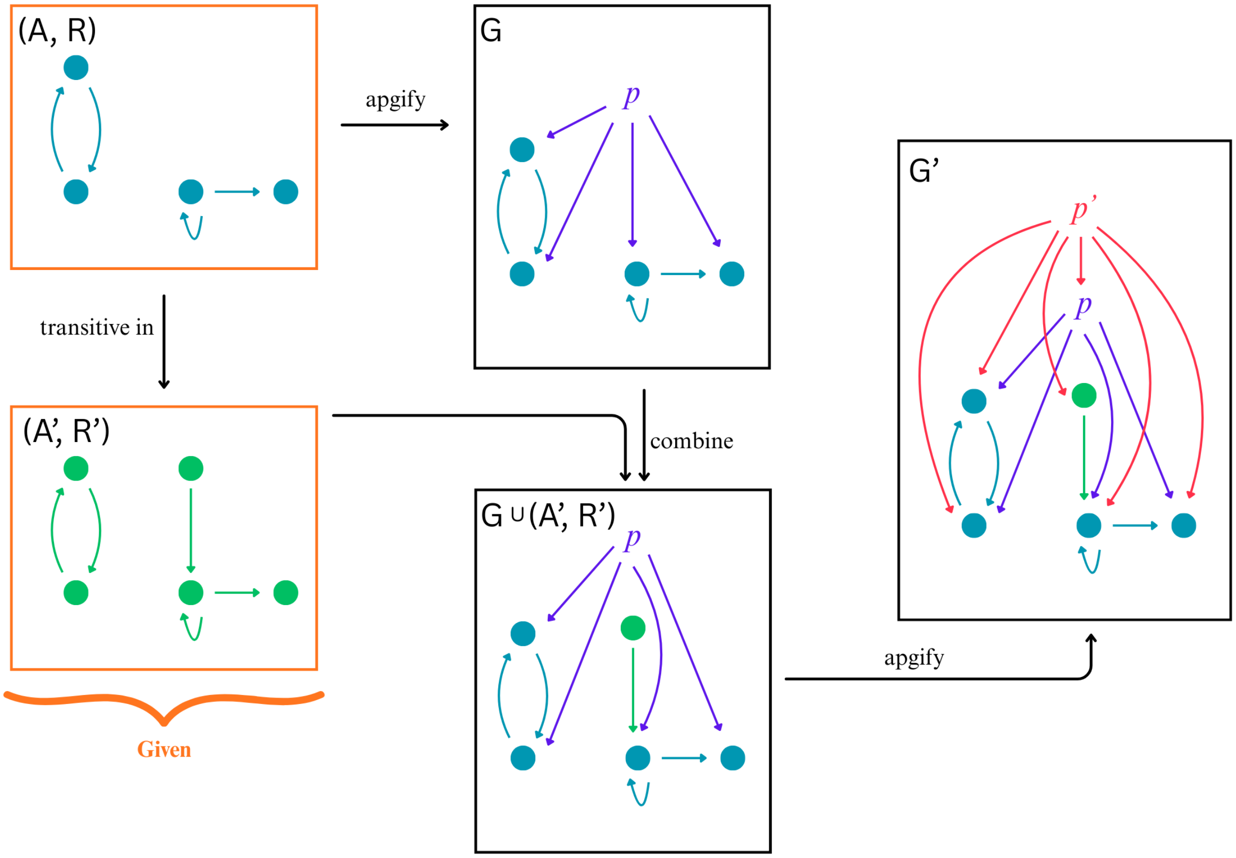
We note that since (A', R') is extensional, it must be that (A, R) is also extensional. By Lemma 5.3.3 we may apg-ify (A, R) into an extensional apg G with point p. We will want to combine this new graph G with (A', R') to produce another extensional relation, and then apg-ify this new larger relation.

Formally, define a graph G^* where $V(G^*) = V(G) \cup A'$ and $E(G^*) = E(G) \cup \text{conv}(R')$, where conv($R'$) is the converse of R'. Note that if we added a new node to G in the previous step, then G^* is the graph representing (A', R') that also includes this new node. If not, then G^* is simply the graph representing (A', R'). It is possible that in G^* there is some node p^* so that p and p^* have the same children; in this case we remove p from G' to preserve the graph's extensionality.

Finally, by our lemma we can apg-ify G^* to produce an apg G' with point p' so that the collection of descendant subgraphs below elements in the image of G' is isomorphic to G^*. See Figure 5.3.7 below for an example of the apg-ification progress on a given (A, R) and (A', R').

Now for the payoff! We note that G is an exact picture with an injective decoration. Additionally, by assumption $(A, R) \cong (B, \in)$ and so there is an injective decoration d assigning nodes of G to elements of B. Additionally, G is a descendant subgraph of exact picture G', since $G'[p] = G$. Therefore by BA_2 there is an extension of d to a new decoration d' which assigns the nodes of G' to elements of some transitive set B^*. Since this decoration is an extension of d, it follows that $B \subseteq B^*$.

Finally, we can define the set $B' = \{b \in B^* \mid \exists a \in A' \text{ with } d'(b) = a\}$. That is, B' is the subset of B^* containing only the elements which map to elements of A' by d'. Since (A', R') is transitive, we know that B' must be transitive. Thus we have produced a set B' with the desired properties, and our equivalence is proven. ■



☆**Figure 5.3.7:** A diagram describing the process of constructing apgs G and G' given example sets and relations (A', R') and (A, R).

6 COMPARING THE ANTI-FOUNDATION AXIOMS

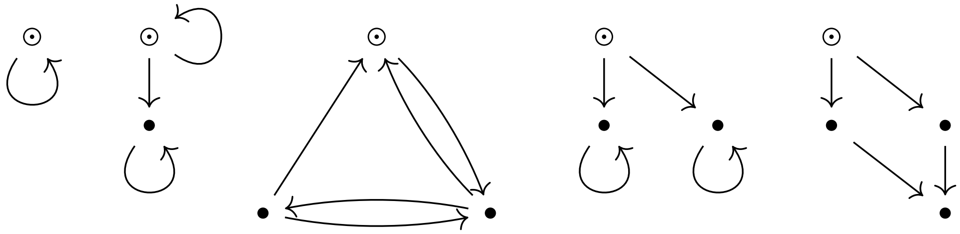
We've seen from Theorems 4.3.6, 4.4.6, and 5.1.1 that if G is an apg, then

$$\text{G is strongly extensional} \Longrightarrow \text{G is Scott-extensional} \Longrightarrow \text{G is Finsler-extensional} \Longrightarrow \text{G is extensional}$$

so therefore

$$\text{G is an exact picture by AFA} \Longrightarrow \text{G is an exact picture by SAFA} \Longrightarrow \text{G is an exact picture by FAFA} \Longrightarrow \text{G is an exact picture by BAFA}$$

Additionally, we've seen examples of apgs (from Figures 4.1.3, 4.3.3, 4.4.4, 3.4.5, and 3.4.1) which show that apgs are not necessarily extensional, and none of the above implication arrows flow the other direction.



☆**Figure 6.1.1**: From left to right: (1) An apg that is strongly extensional, (2) an apg that is Scott-extensional but not strongly extensional, (3) an apg that is Finsler-extensional but not Scott-extensional, (4) an apg that is extensional but not Finsler-extensional, and (5) an apg that is not extensional.

Thus in a sense these axioms form a hierarchy of extensions of the standard ZFC universe. If V represents the universe of ZFC sets, V_{AFA} represents the universe of all ZFA sets, and so on, then

$$V \subset V_{\text{AFA}} \subset V_{\text{SAFA}} \subset V_{\text{FAFA}} \subset V_{\text{BAFA}}.$$

7 RIEGER’S THEOREM

This concluding chapter will be an exploration and proof of Rieger’s Theorem, which is an important tool utilized when constructing models of ill-founded set theories. Essentially, Rieger’s theorem states that every structure satisfying a certain “fullness” condition obeys the axioms of $\mathrm{ZFC}^{-\mathrm{f}}$. Rieger’s Theorem was first presented by Ladislav Rieger in [15], but we will follow the formulation of the theorem presented in Appendix B of [1].

7.1 Systems

To present Rieger’s theorem, we’ll want to expand our definition of a graph to allow us to work with a proper class of nodes; that is, a collection of nodes which is too “large” to itself be a set.

❀**Definition: 7.1.1** A system is a class M of nodes with a corresponding class E of (directed) edges, which are ordered pairs of nodes. That is, $E \subseteq M \times M$, and we do not recognize multi-edges. As before, we write $a \to b$ to mean $(a, b) \in E$, and say that b is a child of a. Lastly, for M to be a system we require that for each node $a \in M$ the collection of children of a must be a *set* (not a proper class). We denote this set as $\mathrm{chil}(a) = \{b \in M \mid a \to b\}$.

We’ll refer to systems using the name of their nodes class; that is the system above would be referred to as simply M.

❏ **Example 7.1.2:** Any graph in the usual sense is an example of a system. Since the collection of nodes all nodes forms a set, we could call this a small system.

❏ **Example 7.1.3:** The von Neumann Universe V (that is, the collection of all ZFC sets) forms a system, where the edge relation is given by set membership; that is $a \to b$ whenever $b \in a$.

Because V is a proper class, we might call this a large system.

✿**Definition 7.1.4:** We say that a system M is full if for each subset of nodes $X \subseteq M$ there is a unique $a \in M$ such that X is precisely the set of children of a; that is $\mathrm{chil}(a) = X$. We denote this unique $a \in M$ as $\mathrm{par}(X)$.

A full system is essentially a structure with a binary relationship and two important properties. The first is that the binary relation is "set-like", meaning the for any element of the structure, the collection of things that it relates to is a set. The second property is that for any subset of elements of the structure, there is a unique element that relates to all of the elements in that subset and nothing else.

7.2 The Theorem

✧**Theorem 7.2.1 (Rieger's Theorem):** It follows from $\mathrm{ZFC}^{-\mathrm{f}}$ that every full system is a model of $\mathrm{ZFC}^{-\mathrm{f}}$.

Proof. Let M be a full system. We'll list the axioms of $\mathrm{ZFC}^{-\mathrm{f}}$ one by one, and show that M along with the edge relation $\to$ models each one of them.

Extensionality: The Axiom of Extensionality reads:

$$\forall x \forall y [\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y].$$

Suppose we're given $x, y \in M$ and know that for all $z \in M$, $x \to z$ if and only if $y \to z$. Then we know that $\mathrm{chil}(x) = \mathrm{chil}(y)$, call this set of children C. By fullness, there exists only one $a \in M$ with $\mathrm{chil}(a) = C$, and therefore $a = x = y$. Since this implication holds for any $x, y \in M$, we conclude that $M \vDash$ Extensionality.

Pair: The Pair Axiom reads:

$$\forall x \forall y \exists z \forall w[(w \in z) \Rightarrow (w = x \vee w = y)].$$

Let $x, y \in M$. By fullness, there exists $z \in M$ with $\text{chil}(z) = \{x, y\}$. Note that for any $w \in M$, $z \to w$ if and only if $w = x$ or $w = y$. Thus we conclude that $M \vDash \text{Pair}$.

Union: The Union Axiom reads:

$$\forall x \exists y \forall z \forall w[(z \in w \wedge w \in x) \Rightarrow (z \in y)]$$

Let $x \in M$. Suppose w is a child of x, and consider the set $\text{chil}(w)$. By fullness there then exists a node y so that $\text{chil}(y) = \cup\{\text{chil}(w) \text{ such that } w \leftarrow x\}$. Then $x \to w$ and $w \to z$ implies $y \to z$, and so we conclude $M \vDash \text{Union}$.

Power Set: The Power Set Axiom reads:

$$\forall x \exists y \forall z \forall w[z \in y \Leftrightarrow (w \in z \Rightarrow w \in x)]$$

Let $x \in M$ and consider $\text{chil}(x)$. By fullness, for all subsets $A \subseteq \text{chil}(x)$, there exists a unique node z_A whose children are exactly A. Then again by fullness there is a unique node y whose children are precisely the set of all nodes $\{z_A \mid A \subseteq \text{chil}(x)\}$.

This node y will be the powerset of x, since $z \leftarrow y$ if and only if the children of z are all children of x. That is, $(w \leftarrow z)$ implies $(w \leftarrow x)$, and so $M \vDash \text{Power Set}$.

Empty: The Empty Axiom is implied by Infinity and Separation, and thus is technically redundant. Nonetheless, it's straightforward to show directly that $M \vDash \text{Empty}$. The Empty Axiom reads

$$\exists x \forall y \neg(y \in x)$$

We note that $\varnothing \subseteq M$, and thus by fullness there exists a node x so that $x \to y$ is false for any $y \in M$. Thus $M \vDash$ Empty.

Infinity: One formulation of the Infinity Axiom reads

$$\exists z[((\exists x \in z)\forall y \neg(y \in x)) \wedge ((\forall x \in z))(\exists y \in z)(x \in y)]$$

That is, there exists a set z so that the empty set is in z, and for all sets x in z there is another set y in z which contains x.

First we note that the empty set is a subset of M, and so there exists a unique node Δ_0 with no children. We recursively define

$$\Delta_{n+1} = \mathrm{par}(\mathrm{chil}(\Delta_n) \cup \Delta_n))$$

That is, Δ_{n+1} is the unique parent of the children of Δ_n as well as Δ_n itself. Each Δ_n is a node in M, and so by fullness there is a unique node that is the parent of all of them. That is, there exists a node $\Delta_\omega \in M$ so that

$$\Delta_\omega = \mathrm{par}(\{\Delta_n \mid n = 0, 1, 2, \ldots\})$$

We note that $\Delta_\omega \to \Delta_0$, and there is no node x in M so that $\Delta_0 \to x$. Additionally, any node that is a child of Δ_ω is a Δ_n for some n. Then, Δ_{n+1} is also a node of Δ_ω, and $\Delta_{n+1} \to \Delta_n$.

Therefore, Δ_ω satisfies both conditions of the Infinity Axiom, and so $M \vDash$ Infinity.

Schema of Comprehension: Let φ be a single-variable formula in the language of set theory. The Comprehension Axiom schema reads:

$$\forall x \exists y \forall z[(z \in y) \Leftrightarrow (z \in x \wedge \varphi(z)]$$

Let $x \in M$. We can then define the subset $Y \subseteq M$ by

$$Y = \{z \in \text{chil}(x) \text{ such that } M \models \varphi(z)\}$$

By fullness, there exists a unique $y \in M$ so that $\text{chil}(y) = Y$. Then $y \to a$ if and only if $x \to a$ and $M \models \varphi(a)$. Thus we conclude that $M \models$ Comprehension.

Schema of Collection: Following the lead of [1], we'll utilize the Collection Axiom schema in place of the more standard Replacement Axiom schema. In standard ZFC, the two are equivalent, but the standard proof that Collection follows from Replacement makes essential use of Foundation, which is omitted from $\text{ZFC}^{-\text{f}}$. In certain weaker contexts, Collection is a strictly stronger axiom than Replacement; one can see [7] for further discussion on this fact.

Let $\varphi(x, y)$ be a formula in the language of set theory. The Collection Axiom schema reads:

$$\forall a \exists z (\forall x \in a)[(\exists y\, \varphi(x, y)) \Rightarrow ((\exists y \in z)(\varphi(x, y))]$$

That is, for any set a and for any $x \in a$, there exists a set z so that if there is a y satisfying $\varphi(x, y)$, there is a corresponding $y' \in z$ that also satisfies $\varphi(x, y')$.

Morally, the weaker Replacement Axiom schema lets us use a function to transform every element of a set A and receive a new set $F(A)$. Collection lets us use the more general notion of a relation in place of a function: given a set A and a relation R, by Collection we can form a new set $C(A)$ where, for every $a \in A$, there is *at least one* $c \in C(A)$ so that aRc. Each a could relate to many things, so there are many possible choices for c, multiple of which could be included in $C(A)$!

Let's show $M \models$ Collection. Let $\varphi(x, y)$ be a formula in the language of set theory (perhaps using constants from M) with at most x and y free.

Then let $a \in M$, and assume that if $x \to a$ then there exists $y \in M$ so that $M \vDash \varphi(x, y)$. By collection on the *set* $\text{chil}(a)$, there exists a *set* b so that for all $x \in \text{chil}(a)$, there exists $y \in b$ so that $\varphi(x, y)$ holds.

Then $b \cap M$ is a subset of M, so by fullness there exists $c \in M$ so that $\text{chil}(c) = b \cap M$. Then for all x so that $a \to x$, there exists y so that $c \to y$ and $M \vDash \varphi(x, y)$. Thus we conclude that $M \vDash$ Collection.

Choice: Following the presentation in [1], the Choice Axiom reads:

$$\forall a[(\forall x \in a)(\exists y(y \in x)) \ \wedge \ (\forall x_1, x_2 \in a)[\exists y(y \in x_1 \wedge y \in x_2) \Rightarrow x_1 = x_2]]$$

$$\Downarrow$$

$$\exists z(\forall x \in a)(\exists y \in x)(\forall u \in x)[u \in z \Leftrightarrow u = y]$$

Our first line merely assumes that a is a set of non-empty disjoint sets. If this is the case, our implication tells us there exists a set z containing an element y "chosen" from each set x contained in a.

Let a be a node in M satisfying the hypothesis of Choice (the top line). That is, for every x so that $a \to x$, there exists a child y so that $x \to y$. Additionally, no children of a have any children in common; if x_1 and x_2 are children of a with $x_1 \to y$ and $x_2 \to y$, then $x_1 = x_2$.

We can then consider the *set* $\{\text{chil}(x) \mid a \to x\}$, and by Choice produce a set S containing one element chosen from each set $\text{chil}(x) \in S$.

By fullness, there then exists a unique node z so that $\text{chil}(z) = S$. Then for any node x so that $a \to x$, there is a y that is a child of x that is also a child of z. Therefore z is the parent of one child "chosen" from each of the children of children of a. Thus $M \vDash$ Choice.

Thus we conclude that M is a model of ZFC$^{-\mathrm{f}}$. ■

7.3 Applications of the Theorem

Rieger's Theorem is often utilized to construct models of non-foundational set theories. All four of the axioms we've introduced in previous sections (AFA, SAFA, FAFA, and BAFA) are anti-foundation axioms which are appended to ZFC$^{-\mathrm{f}}$. When we want to prove that a certain construction is a model of one of these set theories, Rieger's theorem allows us to verify that the structure is full and conclude that it then satisfies the axioms of ZFC$^{-\mathrm{f}}$. Then all that remains to show is that the structure satisfies our chosen anti-foundation axiom, and we've proven that it is a model of our theory. Thus, Rieger's theorem saves a lot of "legwork" in constructing such models.

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