Non-Foundational Set Theory & Superuniversality

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The Axiom of Foundation

The axioms of ZFC include the Foundation Axiom, which reads

$$\forall x \exists y[(x = \varnothing) \vee ((y \in x) \wedge (x \cap y = \varnothing))]$$

"If x is a non-empty, there is a $y \in x$ so that x and y share no elements."

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Theorem

Foundation implies there is no infinite membership chain $x_0 \ni x_1 \ni x_2 \ni \ldots$

In particular, by Foundation there is no set x so that $x \in x$.

We note that Foundation is neither necessary nor sufficient to prevent Russell-like paradoxes!

Sets as Graphs

It's often useful to depict sets as graphs when discussing set theories without Foundation. We'll use directed graphs with a distinguished node called the point where every other node can be reached by a finite path starting from the point. Such graphs are called APGs.

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Example

We can depict the von Neumann ordinal 2 with either of the following APGs.

$$\begin{tikzcd} \odot \arrow[d] \arrow[dr] & \\ \bullet & \bullet \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} \odot \arrow[d] \arrow[dr] & & \\ \bullet & \bullet \arrow[r] & \bullet \end{tikzcd}$$

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\qquad\qquad
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\end{tikzcd}
$$

If distinct nodes in our graph G represent distinct sets, we say that G is an <u>exact</u> picture. The graph on the left is an exact picture of 2, the graph on the right is not.

Exact Pictures and Non-Well-Founded Sets

Theorem (Mostowski Collapse)

An APG with no infinite path is the picture of exactly one set.

We can always determine if such an APG is an exact picture of the unique set it depicts. But cyclic APGs are a bit more complicated!

Exact Pictures and Non-Well-Founded Sets

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Example

Consider the following APG, which we'll call Q.

$$\begin{tikzcd} \bullet \arrow[loop below] & \odot \arrow[l] \arrow[r] & \star \arrow[loop below] \end{tikzcd}$$

Can Q be an exact picture? The answer depends on whether $\bullet$ and $\star$ can represent distinct sets. But Extensionality tells us $\bullet = \star$ iff $\bullet = \star$.

A Non-Foundational Axiom

[Acz88] puts forth the following axiom as a replacement for Foundation, and we denote the resulting system $\mathsf{ZFC}^{-\mathsf{f}}+$ AFA.

Axiom (Anti-Foundation Axiom)

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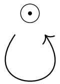
So, the following APG depicts a set in our universe of discourse!



This is a picture of the set $\Omega = \{\Omega\} = \{\{\Omega\}\} = \{\{\ldots\}\}$.

In $\mathsf{ZFC}^{-\mathsf{f}} +$ AFA, our previous example Q is <u>not</u> an exact picture; in fact, it's another picture of Ω.

Another Non-Foundational Axiom

[Acz88] also suggest a different replacement for Foundation, called Boffa's Anti-Foundation Axiom (BAFA) or the Superuniversality Axiom.

Definition

We say an APG G is extensional when distinct nodes of G have distinct collections of children.

Axiom (weak BAFA)

Every extensional APG is an exact picture.

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$\mathsf{ZFC}^{-\mathsf{f}} + \mathsf{BAFA}$ implies the existence of a set $\Omega = \{\Omega\}$. However it also recognizes the distinct set $Q = \{\bullet, \star\}$ where $\bullet = \{\bullet\}$ and $\star = \{\star\}$.

Kunen's Theorem fails in $\mathsf{ZFC}^{-\mathsf{f}} + \mathsf{BAFA}$, in which there are non-trivial elementary embeddings of V to itself (see discussion in [DGHJ]).

Thanks!

Thanks to MAMLS for the opportunity to present, and thank you for your attention!

References:

[Acz88] Peter Aczel. Non-Well-Founded Sets. Center For the Study of Language and Information, Stanford University, 1988.

[DGHJ] Ali Sadegh Daghighi, Mohammad Golshani, Joel David Hamkins, and Emil Jeřábek. The Foundation Axiom and Elementary Self-Embeddings of the Universe. 2014.